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A planar topological horseshoe theory with applications to computer verifications of chaos

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Abstract

In this paper, we present an elementary theory on the existence and robustness of horseshoes under perturbations in terms of stability of crossing. The framework is developed in the setting of 2D Euclidean space but can be generalized to metric spaces. As an application, we give a rigorous verification of the existence of a horseshoe in the Ikeda map.

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1. Introduction

It is well accepted that the existence of a 'horseshoe' embedded in a dynamical system should be the most compelling signature of chaos, both in dissipative and conservative systems, and the horseshoe theory (Smale horseshoe or current topological horseshoe theory) with symbolic dynamics provides a powerful tool in rigorous studies of complicated dynamics such as chaos in dynamical systems. Since the first result on Smale horseshoes, remarkable progress has been made in seeking sufficient conditions for the existence of horseshoes in dynamical systems, both in discrete time and continuous time (by virtue of Poincaré map) cases [1–15].

However, there still remains much work to do in finding more appropriate approaches to the existence of horseshoes in dynamical systems, especially in finding approaches suitable for computer study of chaotic dynamical systems.

In this paper, we present an elementary theory on the existence of horseshoes and their robustness of under perturbations in terms of stability of crossing. We do not use the well-known Conley index theory to study horseshoes, because it requires much topological background, and there are excellent papers on this topic [4–8]. We hope to present a theory that is rigorous but nevertheless understandable to readers with less topological knowledge. Without loss of generality, we only consider the situation where the metric space is R^2 for the reader's convenience, because the theory developed in the setting of the planar case can be restated in the context of n-dimensional space R^n or metric space, and proved in the same manner.

2. Preliminaries

First we recall some aspects of symbolic dynamics.

Let $S_m = \{0, 1, \ldots, m-1\}$ be the set of non-negative successive integers from 0 to $m-1$. Let Σ_m be the collection of all bi-infinite sequences or one-sided sequences with their elements from S_m, i.e., every element s of Σ_m is of the following form,

$$s = \{\ldots, s_{-n}, \ldots, s_{-1}, s_0, s_1, \ldots, s_n, \ldots\}, \qquad s_i \in S_m,$$

or

$$s = \{s_0, s_1, \ldots, s_n, \ldots\}, \qquad s_i \in S_m.$$

Now consider another sequence $\bar{s} \in \Sigma_m$

$$\bar{s} = \{\ldots, \bar{s}_{-n}, \ldots, \bar{s}_{-1}, \bar{s}_0, \bar{s}_1, \ldots, \bar{s}_n, \ldots\}, \qquad \bar{s}_i \in S_m,$$

or

$$\bar{s} = \{\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_n, \ldots\}, \qquad \bar{s}_i \in S_m.$$

The distance between s and $\bar{s}$ is defined as

$$d(s, \bar{s}) = \sum_{-\infty}^{\infty} \frac{1}{2^{|i|}} \frac{|s_i - \bar{s}_i|}{1 + |s_i - \bar{s}_i|},$$

in the case of bi-infinite sequences, or

$$d(s, \bar{s}) = \sum_{i=0}^{\infty} \frac{1}{2^{i}} \frac{|s_i - \bar{s}_i|}{1 + |s_i - \bar{s}_i|}, \tag{2.1}$$

in the case of one-sided sequences.

With the distance defined as (2.1), Σ_m is a metric space, and the following facts are well known [9].

Proposition 2.1. *The space Σ_m is*

(i) compact
(ii) totally disconnected
(iii) perfect.

A set having the three properties in the above proposition is often defined as a Cantor set, such a Cantor set frequently appears in characterization of complex structure of invariant set in a chaotic dynamical system.

Furthermore, now define a m-shift map $\sigma : \Sigma_m \to \Sigma_m$ as follows:

$$\sigma(s)_i = s_{i+1}.$$

Then there are the following results.

Proposition 2.2. *(a) $\sigma(\Sigma_m) = \Sigma_m$ and σ is continuous. (b) The shift map σ as a dynamical system defined on Σ_m has the following properties:*

(i) σ has a countable infinity of periodic orbits consisting of orbits of all periods;
(ii) σ has an uncountable infinity of nonperiodic orbits and
(iii) σ has a dense orbit.

For a proof of the above statements, we refer the reader to [9]. A consequence of statement (b) is that the dynamics generated by the shift map σ is sensitive to initial conditions, and therefore is chaotic.

Next we recall the semi-conjugacy in terms of a continuous map and the shift map σ, which is conventionally defined as follows.

Definition 2.1. *Let X be a compact metric space, and $f : X \to X$ a continuous map. If there exists a continuous and onto (surjective) map*

$$h : X \to \Sigma_m$$

such that $h \circ f = \sigma \circ h$, then f is said to be semi-conjugate to σ.

For the reader's convenience, we recall the concept of topological entropy as follows.

Definition 2.2. *Let X be a compact metric space, and $f : X \to X$ a continuous map. A finite set $E \subset X$ is called (n, ε)-separated if for every two different points $x, y \in E$, there exists $0 \leqslant j < n$ such that the distance between $f^j(x)$ and $f^j(y)$ is greater than ε. Now, let the number $s(n, \varepsilon)$ denote the cardinality of a maximum (n, ε)-separated set:*

$$s(n, \varepsilon) = \max\{\text{card } E : \ E \text{ is } (n, \varepsilon)\text{−separated}\}.$$

The topological entropy of the map f is defined as

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon).$$

Proposition 2.3. *Let X be a compact metric space, and $f : X \to X$ a continuous map. If there exists an invariant set $\Lambda \subset X$ such that $f|\Lambda$ is semi-conjugate to the m-shift σ, then*

$$h(f) \geqslant h(\sigma) = \log m,$$

where $h(f)$ denotes the entropy of the map f. In addition, for every positive integer k,

$$h(f^k) = kh(f).$$

The details of topological entropy can be found in many books on dynamical systems (for example, [16]).

3. Topological horseshoe theorems

To develop the main theory in this section, we first give some notions and notations.

Let R^2 be the 2D Euclidean space. Let D be a compact set of R^2, and D_i, $i = 1, 2, \ldots, m$, be compact subsets (usually quadrangles) of D homeomorphic to the unit square or unit disc. Let ∂D_i be the boundary of D_i. Let $f : D \to X$ be a piecewise continuous map which is continuous on each compact set D_i. We introduce some concepts and notations as follows.

For each D_i, $1 \leqslant i \leqslant m$, let d_i^1 and d_i^2 be two fixed disjointed arcs contained in the boundary ∂D_i. A connected subset l of D_i is said to connect d_i^1 and d_i^2, if $l \cap d_i^1 \neq \emptyset$ and $l \cap d_i^2 \neq \emptyset$, and we denote this by $d_i^1 \stackrel{l}{\longleftrightarrow} d_i^2$.

Definition 3.1. *Let $l \subset D_i$ be a connected subset, we say that $f(l)$ is crossing over D_j, if l contains a connected subset l' such that $f(l') \subset D_j$, $f(l') \cap D_j^1 \neq \emptyset$ and $f(l') \cap D_j^2 \neq \emptyset$, i.e., $d_j^1 \stackrel{f(l')}{\longleftrightarrow} d_j^2$. In this case, we denote it by $f(l) \mapsto D_j$. In case that $f(l) \mapsto D_j$ holds true for every connected subset $l \subset D_i$ satisfying $d_i^1 \stackrel{l}{\longleftrightarrow} d_i^2$, we say that $f(D_i)$ is crossing over D_j,* *and denote it by $f(D_i) \mapsto D_j$ in terms of two pairs $\left(d_i^1, d_i^2\right)$ and $\left(d_j^1, d_j^2\right)$. In the following, we only say $f(D_i) \mapsto D_j$ for convenience, and speak of the two pairs $\left(d_i^1, d_i^2\right)$ and $\left(d_j^1, d_j^2\right)$ in case of confusion arising.*

Example 3.1. Let

- $D_1 = \{(x, y) \in R^2 : 0 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}$, with
- $d_1^1 = \{(x, y) \in R^2 : x = 0, -1 \leqslant y \leqslant 1\}$ and $d_1^2 = \{(x, y) \in R^2 : x = 1, -1 \leqslant y \leqslant 1\}$, and
- $D_2 = \{(x, y) \in R^2 : 2 \leqslant x \leqslant 3, y = 0\}$ with
- $d_2^1 = \{(x, y) \in R^2 : x = 2, y = 0\}$ and $d_2^2 = \{(x, y) \in R^2 : x = 3, y = 0\}$.

Consider a map f with

$$f|_{D_1} : f(x, y) = (4x, 0)^{\mathrm{T}}$$
$$f|_{D_2} : f(x, y) = (4(x-2), 0)^{\mathrm{T}}.$$

It is easy to see that $f(D_i) \mapsto D_j$, $1 \leqslant i, j \leqslant 2$. Note that D_1 is a rectangle and D_2 is just a section of line.

Le X be a metric space, D is a compact subset of X. In the following, we consider compact subsets $D_1, \ldots, D_{m-1}$. and D_m contained in D, and assume $f : D \to X$ is a piecewise continuous map defined as above. To present a more general result on horseshoes for piecewise continuous maps, let us first recall an established fact.

Definition 3.2. *Let γ be a compact connected subset of D, such that for each $1 \leqslant i \leqslant m$, $\gamma_i = \gamma \cap D_i$ is non-empty and compact, then γ is called a connection with respect to $D_1, \ldots, D_{m-1}$ and D_m.*

Let F be a family of connections γs with respect to $D_1, \ldots, D_{m-1}$, and D_m satisfying the following property:

$$\gamma \in F \Rightarrow f(\gamma_i) \in F.$$

Then F is said to be a f-connected family with respect to $D_1, \ldots, D_{m-1}$, and D_m.

Lemma 3.1 (Horseshoe lemma). *Suppose that there exists a f-connected family F with respect to $D_1, \ldots, D_{m-1}$ and D_m. Then there exists a compact invariant set $K \subset D$, such that $f|K$ is semi-conjugate to m-shift dynamics.*

For a proof of this lemma, see [12]. Now we have the following result

Theorem 3.1. *Suppose that the map $f : D \to R^2$ satisfies the following assumptions:*

(1) There exist m mutually disjoint subsets $D_1, \ldots$, and D_m of D, the restriction of f to each D_i, i.e., $f|D_i$ is continuous.
(2) The relation $f(D_i) \mapsto D_j$, holds for $1 \leqslant i, j \leqslant m$.

Then there exists a compact invariant set $K \subset D$, such that $f|K$ is semi-conjugate to m-shift map, and

$$h(f) \geqslant \log m.$$

Proof. It is enough to show that there exists a f-connected family with respect to $D_1, \ldots, D_{m-1}$ and D_m in view of lemma 3.1. To this end, let F be the family of arcs crossing over every $D_1, \ldots, D_{m-1}$ and D_m : $l \in F$ if and only if $l \subset D$ and there exists a subset $l_i \subset l$ such that $d_i^1 \overset{l_i}{\longleftrightarrow} d_i^2$. It is clearly non-empty. Now we show that F is a

f-connected family with respect to $D_1, \ldots, D_{m-1}$ and D_m. For $l \in F$, $l \cap D_i \supset l_i$, which implies that

$$f(l \cap D_i) \supset f(l_i).$$

Since $f(l_i)$ is crossing over $D_1, \ldots, D_{m-1}$ and D_m by condition (2), so is $f(l \cap D_i)$, therefore

$$f(l \cap D_i) \in F,$$

showing that F is a f-connected family with respect to $D_1, \ldots, D_{m-1}$ and D_m. □

4. Stability of crossing

It is well known that the existence of horseshoes in a dynamical system is often demonstrated through computer computations and simulations, and this is usually the case for the Poincaré map derived from ordinary differential equations. In doing so, one has unavoidable round-off and some computation errors, thus obtaining an approximate map for the nominal Poincaré map. Therefore, it is more practical to provide a method for existence of horseshoes in a map by means of its approximate map, thus offering a criterion to ensure validity and rigour of computer-simulation arguments for horseshoes in dynamical systems. In this regard, it is necessary to study stability of crossing, i.e., whether the 'crossing over' can be preserved under (small) perturbations of map, which is equivalent to a robustness of the crossing property or stability of crossing in the following sense.

Definition 4.1. *Let A and B be two compact subsets of D, and a continuous map $f : D \to R^2$ with the property $f(A) \mapsto B$. If there exists an $\delta > 0$ such that every continuous map $g(A) \to R^2$ with $\|g - f\| = \min_{x \in A} \|g(x) - f(x)\| < \delta$ satisfies $g(A) \mapsto B$, then the crossing of $f(A)$ over B is said to be stable.*

Now we see under what conditions the crossing can be stable. For a compact domain D homeomorphic to the unit square with two fixed arcs d^1 and d^2, consider a subset $B \subset R^2$ with the following properties:

(a) $B = B_1 \cup B_2 \cup B_3$ with $D \cap B = B_3$ homeomorphic to unit square.
(b) There exist two subarcs $\alpha \subset \mathrm{int}(d^1)$ and $\beta \subset \mathrm{int}(d^2)$ such that $\alpha, \beta \subset B_3$, and $B_3 \cap B_1 = \alpha$ and $B_3 \cap B_2 = \beta$, where $\mathrm{int}(d)$ denotes the interior of the arc d, i.e., the arc d with its end points taken off.
(c) $B_1 - \alpha \subset R^2 - D$, $B_2 - \beta \subset R^2 - D$.
(d) $\partial B_3 - \alpha \cup \beta \subset \mathrm{int}\, D$.

A subset B with these properties is called a companion set of D with respect to the fixed arcs d^1 and d^2, or just a companion set of D for brevity.

The following fact is obvious.

Proposition 4.1. *Consider a compact domain D homeomorphic to unit square with two fixed arcs d^1 and d^2. And $B = B_1 \cup B_2 \cup B_3$ be its companion set. Suppose a compact domain C with fixed arcs c^1 and c^2 satisfies $C \mapsto B$, then there exists an $\varepsilon > 0$ such that for any continuous map $g : C \to R^2$ that satisfies*

$$\|g(x) - x\| < \varepsilon, \qquad \forall\, x \in C,$$

one has

$$g(C) \mapsto D.$$

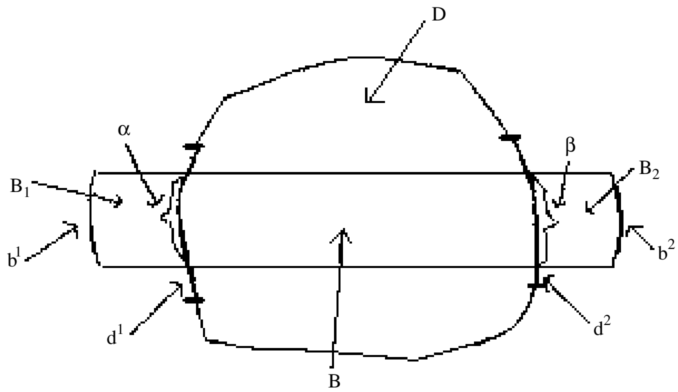

Figure 1. The compact set D and its companion set B.

Proof. This theorem is just a consequence of the following result.

Proposition 4.2. *Consider a compact domain D homeomorphic to unit square with a fixed pair (d^1, d^2). Let $B = B_1 \cup B_2 \cup B_3$ be its accompanied set (see figure 1) satisfying*

$$\mathrm{dist}(\partial D - d^1 \cup d^2, B_1 \cup B_2) \geqslant \eta, \tag{4.1}$$

where $\mathrm{dist}(A, B) = \min_{x \in A, y \in B} \|x - y\|$, *and the fixed pair $b^1 \subset \partial B_1 - \alpha$ and $b^2 \subset \partial B_2 - \beta$ of B satisfies*

$$\mathrm{dist}(b^1, d^1) \geqslant \eta, \qquad \mathrm{dist}(b^2, d^2) \geqslant \eta, \tag{4.2}$$

$$\mathrm{dist}(\partial D - d^1 \cup d^2, B_3) \geqslant \eta, \tag{4.3}$$

$$\mathrm{dist}(d^1, d^2) \geqslant 2\eta, \tag{4.4}$$

$$\mathrm{dist}(B_1, B_2) \geqslant 2\eta. \tag{4.5}$$

Suppose that a compact domain C with fixed pair (c^1, c^2) satisfies $C \mapsto B$, then for any continuous map $g : C \to R^2$ that satisfies

$$\|g(x) - x\| < \eta, \qquad \forall\, x \in C, \tag{4.6}$$

one has

$$g(C) \mapsto D.$$

Proof. It is enough to show that for every connected arc $l \subset C$ connecting the fixed pair (c^1, c^2), the crossing $g(l) \mapsto D$ holds. Now for any such a connected arc, one has $l \mapsto B$ by assumption. This implies that there exist two points $\bar{x}, \tilde{x} \in l$ such that $\bar{x} \in b^1$ and $\tilde{x} \in b^2$. Since $|g(\bar{x}) - \bar{x}| < \eta$, $|g(\tilde{x}) - \tilde{x}| < \eta$, one has $g(\bar{x}) \notin D$ and $g(\tilde{x}) \notin D$ by conditions (4.1) and (4.2). From conditions (4.2)–(4.4), it is easy to see that $g(l) \cap (D - \partial D) \neq \emptyset$ due to the continuity of g. To see this, consider the subline of $\bar{l} = l \cap B_3$. In view of condition (4.4), there exists a point $p \in \bar{l}$, such that $\mathrm{dist}(p, d^2) \geqslant \eta$, and $\mathrm{dist}(p, d^1) \geqslant \eta$. From condition (4.2), it is easy to see that $\mathrm{dist}(p, \partial D) \geqslant \eta$. It follows that $g(p) \in (D - \partial D)$, and this is what we want.

Now because of the continuity of g, there is a point $x_1 \in l[\bar{x}, p]$, and a point $x_2 \in l[p, \tilde{x}]$ such that $g(x_1) \in \partial D$ and $g(x_2) \in \partial D$. Here, $l[a, b]$ designates the closed subline of l with end points a and b. It is apparent that $g(x_1) \in d^1 \cup d^2$ and $g(x_2) \in d^1 \cup d^2$. This can be proved as follows. Take the first assertion $g(x_1) \in d^1 \cup d^2$ as an example. Suppose that this is not the case, then one would have $g(x_1) \in \partial D - d^1 \cup d^2$. However, conditions (4.1)–(4.3) mean that $\|g(x_1) - x_1\| \geqslant \eta$, in contradiction to condition (4.4). Furthermore, in view of condition (4.5), it can be seen that there is at least a point $\bar{x}_1 \in l[\bar{x}, p]$, and a point $\bar{x}_2 \in l[p, \tilde{x}]$ such that $g(\bar{x}_1) \in d^1$ and $g(\bar{x}_2) \in d^2$. If this is not the case, then without loss of generality, suppose that every point z satisfying $g(z) \in d^1 \cup d^2$ is contained in d_2. This implies that that there exists a point $q \in l[\bar{x}, p]$ with $g(q) \in d^2$ such that $g(x) \notin D$ for $\forall x \in l[\bar{x}, q)$, where $l[\bar{x}, q)$ designates the semi-closed subline of l with end points q and $\bar{x}$ but not containing the end point q. Note that $|g(\bar{x}) - \bar{x}| < \eta$ implies the inequality $\mathrm{dist}(g(\bar{x}), B_1) < \eta$. It follows from inequality (4.5) that there exists a point $s \in l(\bar{x}, q)$ such that $\mathrm{dist}(g(s), B_1) > \eta$ and $\mathrm{dist}(g(s), B_2) > \eta$. Since $g(s) \notin D$, then we have $\mathrm{dist}(g(s), B) > \eta$. On the other hand, it is easy to see that $s \in B$, which implies that $\mathrm{dist}(g(s), s) > \eta$ and this is in contradiction to the condition (4.6). Therefore, there is at least a point $\bar{x}_1 \in l[\bar{x}, p]$, and a point $\bar{x}_2 \in l[p, \tilde{x}]$ with $g(\bar{x}_1) \in d^1$ and $g(\bar{x}_2) \in d^2$ such that $g(l[\bar{x}_1, \bar{x}_2])$ connects d^1 and d^2, i.e.,

$$d^1 \overset{g(l[\bar{x}_1, \bar{x}_2])}{\longleftrightarrow} d^2.$$

We see that the crossing $g(l) \mapsto D$ holds, therefore $g(C) \mapsto D$. □

In the same way, the following important fact can be proved.

Proposition 4.3. *Consider a compact domain D homeomorphic to the unit square with a fixed pair* (d^1, d^2)*. Let* $B = B_1 \cup B_2 \cup B_3$ *be its accompanied set satisfying*

$$\mathrm{dist}(\partial D - d^1 \cup d^2, B_1 \cup B_2) \geqslant \eta, \tag{4.7}$$

where $\mathrm{dist}(A, B) = \min_{x \in A, y \in B} \|x - y\|$, *and the fixed pair* $b^1 \subset \partial B_1 - \alpha$ *and* $b^2 \subset \partial B_2 - \beta$ *of B satisfies*

$$\mathrm{dist}(b^1, d^1) \geqslant \eta, \qquad \mathrm{dist}(b^2, d^2) \geqslant \eta, \tag{4.8}$$

$$\mathrm{dist}(\partial D - d^1 \cup d^2, B_3) \geqslant \eta, \tag{4.9}$$

$$\mathrm{dist}(d^1, d^2) \geqslant 2\eta, \tag{4.10}$$

$$\mathrm{dist}(B_1, B_2) \geqslant 2\eta. \tag{4.11}$$

Let C be a compact domain with fixed pair (c^1, c^2)*. Suppose that a continuous map* $f : C \to B$ *satisfies* $f(C) \mapsto B$, *then for any continuous map* $g : C \to R^2$ *that satisfies*

$$\|g(x) - f(x)\| < \eta, \qquad \forall\, x \in C, \tag{4.12}$$

one has

$$g(C) \mapsto D.$$

Now in view of the above arguments, it is easy to obtain the following main results in this paper.

Theorem 4.1. *Let D be a compact set of* R^2, *and* D_i, $i = 1, 2, \ldots, m$, *be compact subsets (usually quadrangles) of D, and* $f : D \to R^2$ *be a continuous map. If there exists a companion set* B^i *for each* D_i, *such that* $f(D_i) \mapsto B^j$ *holds for* $i, j = 1, 2, \ldots, m$, *then the crossing*

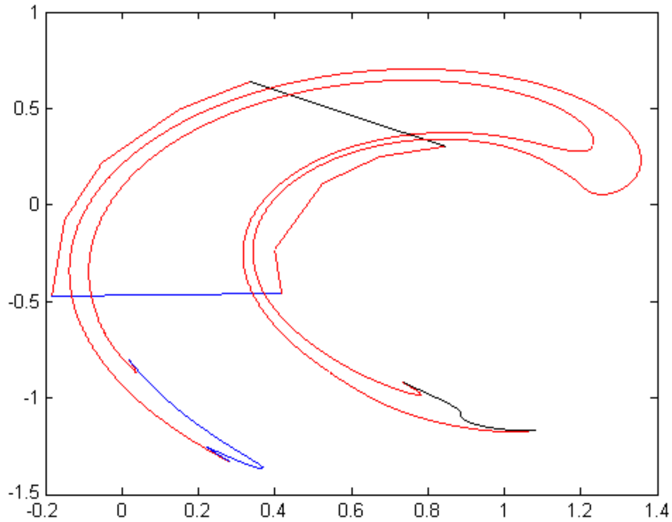

Figure 2. The decagon and its image under three iterations of the Ikeda map.

$f(D_i) \mapsto D_j$ is stable. Therefore there exists a $\delta > 0$ such that for every $g : D \to R^2$ satisfying $\|g - f\| = \min_{x \in A} \|g(x) - f(x)\| < \delta$, there exists a compact invariant set $K \subset D$, such that $f|K$ is semi-conjugate to the m-shift map.

Theorem 4.2. *Let D be a compact set of R^2, and D_i, $i = 1, 2, \ldots, m$, be compact subsets (usually quadrangles) of D, and $f : D \to R^2$ be a continuous map. If there exists a companion set B^i for each D_i, such that $f(D_i) \mapsto B^j$ holds for $i, j = 1, 2, \ldots, m$, supposing that every companion set satisfies conditions (4.7)–(4.11) as in proposition 4.3, then the crossing $f(D_i) \mapsto D_j$ is stable, and for the number $\eta > 0$ such that for every $g : D \to R^2$ satisfying $\|g - f\| = \min_{x \in D} \|g(x) - f(x)\| < \eta$, there exists a compact invariant set $K \subset D$, such that $g|K$ is semi-conjugate to the m-shift map.*

5. Applications to a 2D map

As an application of the main result in this paper, we consider the well-known Ikeda map [1], which is of the following form:

$$x(i+1) = p + \beta(x(i)\cos(t(i)) - y(i)\sin(t(i)))$$
$$y(i+1) = \beta(x(i)\sin(t(i)) + y(i)\cos(t(i)))$$

where

$$t(i) = t(x(i), y(i)) = \kappa - \alpha/(1 + (x(i))^2 + (y(i))^2),$$
$$p = 1, \beta = 0.9, \kappa = 0.4, \alpha = 6.$$

By means of careful computation, we find a decagon with its vertices: (0.3327, 0.6382), (0.1484, 0.4956), (−0.0507, 0.2237), (−0.1502, −0.0782), (−0.1834, −0.4726), (0.4175, −0.4583), (0.3991, −0.2325), (0.5244, 0.1096), (0.6756, 0.2500), (0.8452, 0.3026).

Figure 2 is the image of the decagon and its image under the third iteration of the Ikeda map.

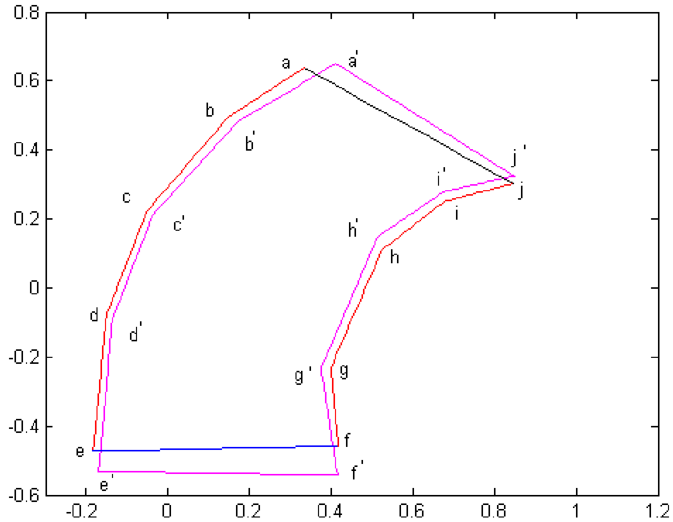

Figure 3. The decagon and its companion set.

For convenience, we denoted the decagon with, in terms of its vertices, say,

$$D = abcdefghij,$$

and construct its companion set (decagon) $B = a'b'c'd'e'f'g'h'i'j'$ as follows as shown in figure 3.

The companion decagon has the vertices: (0.4116, 0.6498), (0.1718, 0.4828), (−0.0358, 0.2138), (−0.1365, −0.0923), (−0.1687, −0.5314), (0.4156, −0.5422), (0.3753, −0.2314), (0.5144, 0.1489), (0.6756, 0.2788) and (0.8489, 0.3251).

Now construct two disjoint hexagons; the upper one is denoted as D_1, with its six vertices being: (0.3327, 0.6382), (0.1484, 0.4956), (−0.0507, 0.2237), (0.5244, 0.1096), (0.6756, 0.2500) and (0.8452, 0.3026). The lower one is denoted as D_2, with its six vertices being: (−0.061 1475, 0.192 0005), (−0.1502, −0.0782), (−0.1834, −0.4726), (0.4175, −0.4583), (0.3991, −0.2325) and (0.512 4965, 0.077 1005).

It is apparent in this example that the decagon $B = a'b'c'd'e'f'g'h'i'j'$ can play the role of the companion set of both the upper hexagon and the lower one. Computer computation shows that the distance between the edge $abcde$ and edge $a'b'c'd'e'$ (as shown in figure 3) is larger than 0.0086, the distance between the edge $fghij$ and edge $f'g'h'i'j'$ is larger than 0.0086, the distance between the edge ef and edge $e'f'$ is larger than 0.0086, the distance between the edge ja and edge $j'a'$ is also larger than 0.0086.

Now, the computer computation shows the images of the upper hexagon and the lower one under the third iteration of the Ikeda map in figures 4 and 5.

A rounding float error arises when we iterate the initial point in the original decagon's edges for the first time, so the first iteration of the initial point has an error of $e \leqslant 10^{-15}$ for the second iteration; the error may be lessened or enlarged when iteration goes on. In order to estimate errors more precisely, we can estimate the largest Lipschitz coefficient in the region $[-10, 10] \times [-10, 10]$ which is mapped into itself under the Ikeda map as shown in [1], and we find that the largest Lipschitz coefficient is not greater than 2.9725, so the third iteration errors for every point on the edges of the hexagons D_1 and D_2 are not more than 9×10^{-15}. From theorem 4.2 one can easily conclude that the third iteration of D_1 and D_2 under the

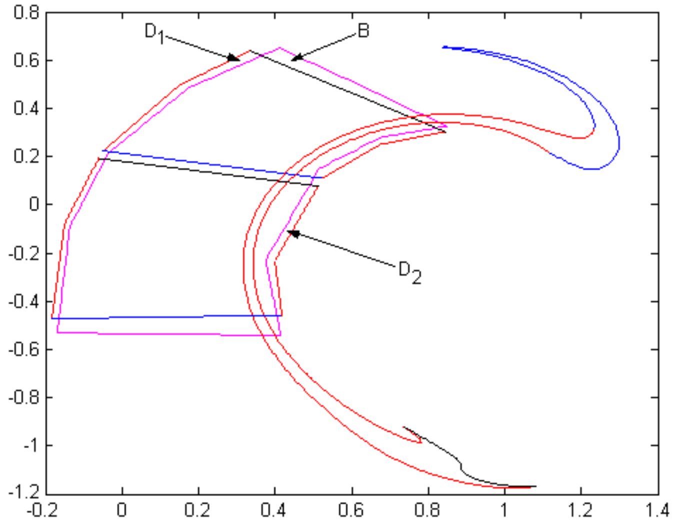

Figure 4. The image of upper hexagon.

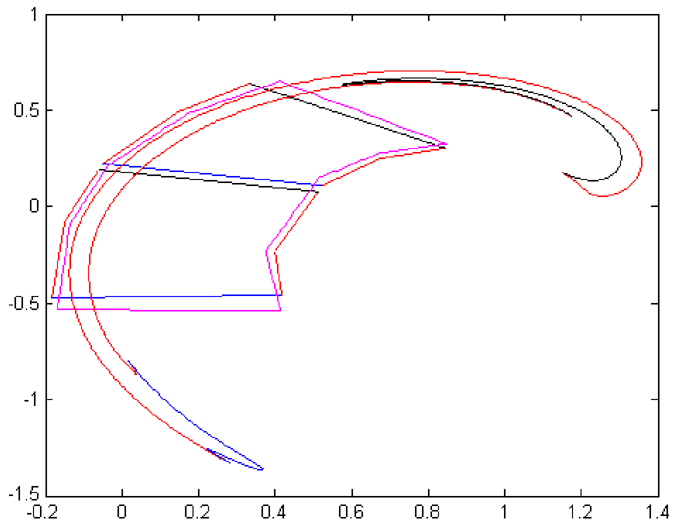

Figure 5. The image of lower hexagon.

Ikeda map has crossing property with respect to D_1 and D_2, thus a horseshoe exists in D for the Ikeda map.

6. Conclusion

In this paper, we have presented a theory on the existence and robustness of horseshoes under perturbations in terms of stability of crossing. The framework is developed in the setting of planar case, but it is easy to see that the main results of this paper can be generalized in the context of n-dimensional space R^n or more general metric space. However, as in [2, 3], the theory developed in this paper is only efficient for dynamics with one-dimensional unstable direction, and we will discuss this problem in a future paper.

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